Jamming of directed traffic on a square lattice

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ABSTRACT

Phase transition from a free-flow phase to a jammed phase is an important feature of traffic networks. We study this transition in the case of a simple square lattice network for different values of data posting rate (ρ) by introducing a parameter p which selects a neighbour for onward data transfer depending on queued traffic. For every ρ there is a critical value of p above which the system become jammed. The $\rho - p$ phase diagram shows some interesting features. We also show that the average load diverges logarithmically as p approaches p_c and the queue length distribution exhibits exponential and algebraic nature in different regions of the phase diagram.

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Key words: Internet traffic, free-flow phase, jammed traffic.

I Introduction

The mechanism of transportation of information is important in different branches of science. The study of highway traffic has been a field of interest for a long time. In view of modern perspectives, the study of internet traffic network has also opened a new field of research. The most important feature of these two networks is a phase transition from a free-flow phase to a conjested phase. The main target in constructing these networks lies in maximising the flow of data and avoiding traffic jam. An extensive study on internet

traffic [1, 2, 3, 4] has already been done. The study of topological structure of internet traffic network [5, 6] has been very helpful in increasing the efficiency of the network. The optimization of traffic flow has also been studied in the recent past [7, 8, 9]. Traffic flow has also been studied on scale-free network [10]. In contrast to such elaborate studies on information transport in realistic models of complex networks (for review, see [11]), a recent study on square lattice by Mukherjee and Manna [12] has also proved to be quite interesting. Although a square lattice is a rather simplified and somewhat unrealistic model of real-life networks, it can nevertheless capture the essential features of traffic flow, such as the occurrence of free flow and the onset of jamming.

The model of Mukherjee and Manna [12] considers an $L \times L$ square lattice and a preferential direction of data flow in $-y$ direction. In a single time step ρL data packets are posted randomly on the sites chosen on the uppermost row. A single data packet is transferred to the lower left (LL) or lower right (LR) neighbour randomly from every site of the row and this process of transferring data continues sequentially for all the rows. Finally, as the data packets reach the lowermost row (considered as sink) they are removed from the system. The average number of data packets per site, $\bar{q}(\rho, t)$ was studied as a function of time for different values of ρ. It was found that for small values of ρ the steady state value and fluctuation of $\bar{q}(\rho, t)$ is small (the 'free-flow' phase) and as ρ is increased, the steady state value of $\bar{q}(\rho, t)$ increases and above a critical value ρ_c, the value of $\bar{q}(\rho, t)$ increases monotonically with time (the 'jammed' phase).

The most important feature in the network considered by Mukherjee and Manna [12] was that, as the value of ρ approaches some critical value ρ_c, the steady state value of $\bar{q}(\rho, t)$ increases rapidly, slowing down the process of data flow. It is due to random selection of one of the neighbours to transfer data packets from each node. In order to investigate this feature in greater detail, we introduce another parameter p, which is the probability by which the more populated node is preferred to a less populated node for onward data transfer. A choice of $p = 0.5$ means random selection between LL and LR site, independent of the population of them, as was done in [12]. As p is gradually decreased from 0.5, we force the data packets to move to the less populated neighbour and thus prevent a particular node from being piled up by data. By decreasing the value

of p, the steady state value of $\bar{q}(\rho, t)$ can be decreased significantly even for high values of ρ below ρ_c, thus increasing the rate of flow through the network. On the other hand, as p is gradually increased the $\bar{q}(\rho, t)$ shows a logarithmic divergence near a critical probability p_c above which the system becomes jammed even for $\rho < \rho_c$. We also observe a lower limit $\rho = \rho_0$, below which the network remains in the free-flow phase, whatever be the value of p.

We have simulated the traffic flow for various combinations of values of ρ and p for three system sizes, $L = 64, 128, 256$. In section II we present the details of the model and in section III the results.

II The Model

The network we have considered is the same as that taken by Mukherjee and Manna [12], together with a newly introduced parameter p as described in section I. Thus, on a square lattice of size $L \times L$ placed on the $x - y$ plane, the lattice sites are the nodes and the lattice bonds are the links. The data flows only along $-y$ direction. Every node can transfer data to any one of the two neighbours, one at the lower left (LL) and the other at the lower right (LR) position.

Each iteration in our simulation comprises of the following three steps : (i) Initially ρL data packets are posted on the randomly chosen nodes of the uppermost row at $y = L$. (ii) The L^2 nodes are updated in helical sequence. Each node transfers a single data packet to the more populated neighbour with probability p and to the less populated neighbour with probability $(1 - p)$. If LL and LR nodes are equally populated, any one of them is chosen randomly for transfer. (iii) Finally, as a data packet reaches the bottom row at $y = 0$, it is removed from the system.

To characterise the flow of traffic, we measure the total number of packets $N(t)$ in the lattice and calculate therefrom the average number of data packets per site $\bar{q}(\rho, t) \equiv N(t)/L^2$ (often called the mean load) at an iteration t. In the so-called free-flow phase, the mean load gets saturated after some time (10^2 to 10^4 iterations depending on the value of p and ρ) indicating that the average fluxes of the inflow and outflow currents

of data packets have reached a steady value and are equal. In the jammed phase the inflow exceeds outflow and the data accumulates continuously in the system, resulting in a monotonically increasing mean load. We have also measured the number of data packets (l) that are in queue at different sites over the lattice (called the queue length) and computed its distribution function $q(l)$.

We have simulated upto 10^5 iterations and averaged over 10 configurations for $L = 64$. For $L = 128$ both the iteration and number of configurations was doubled. But for low values of ρ, near $p = p_c$, the quality of data was poor and we had to iterate upto 3×10^5 time steps and average over 30 configurations. Some runs were repeated also for $L = 256$.

III Results

The study of Mukherjee and Manna [7] was confined to $p = 0.5$ and they found that for a posting rate lower than a critical value $\rho_c = 1$, the system is in a free flow state, while for $\rho > \rho_c$ the system is in a conjested (jammed) state. After introducing the probability p for the selection of recepient node, we observe that the above mentioned result (including the precise value of ρ_c) remains the same for values of $p < 0.5$. Thus, even when we take care to transfer the data *always* to the less populated node, the sytem does get jammed at $\rho > 1$. But as p is increased above 0.5 the value of ρ_c decreases with increase of p.

The effect of introducing p is most significant for high values of ρ $(0.9 < \rho < 1)$. At a posting rate $\rho = 0.98$, the system does get jammed for $p > 0.5$. For values of p just below 0.5 the steady state value of $\bar{q}(\rho, t)$ becomes very high and also fluctuates highly. But if we decrease the value of p, the average load per site decreases (Fig. 1) and also fluctuates much less. Thus by controlling the parameter p we are able to prevent the sites of the lattice from being overloaded, which is very much desired in a traffic network. This trend remains as we decrease ρ further, until the average load per site becomes small for any value of p, and there is no need to control p anymore.

In general for a particular posting rate lower than the critical one ($\rho < \rho_c$) the system will be in a free-flow phase upto a certain critical value of p (say p_c). If we increase the value of p further (select deliberately the more populated node more often) the system

becomes jammed. For different values of ρ, we have determined the value p_c which is obviously a measure of tolerance of the system for 'wrong' selection of node (Fig. 2). The solid line in the figure corresponds to the value of ρ_c for a certain p and similarly the value of p_c for a certain ρ. Just below $\rho = 1$ the value of p_c is $1/2$, but as one decreases ρ, the system needs a larger p to become jammed and the value of p_c increases. This trend continues until at some $\rho = \rho_0$ the value of p_c becomes 1 and cannot increase any more. This value of ρ (which we call lower critical posting rate) is hence the minimum value of ρ below which the system always remains in the free-flow phase even if p is set 1 i.e., the data is deliberately transferred every time to the more populated neighbour. It may be noted that (i) the numerical value of ρ_0 depends on the system size to some extent, it is 0.24 for $L = 64$ and 0.21 for $L = 128$; (ii) the critical probability p_c varies linearly with $\log(\rho - \rho_0)$ (Fig. 3) particularly for $L = 128$. For higher values of ρ, the value of p_c remains same for both system sizes, but for smaller ρ the value of p_c is somewhat smaller for larger size.

As mentioned above, at any posting rate in the region $\rho_0 < \rho < \rho_c$, the system gets jammed and the mean load $\bar{q}(\rho, t = \infty)$ diverges for $p > p_c$. We have observed that for high ($\rho > 0.8$) values of ρ this divergence is logarithmic (Fig. 4),

$$\bar{q}(\rho, t = \infty) = a \log(p_c - p) + b \tag{1}$$

but for $\rho < 0.8$ it is not logarithmic (Fig. 5). For high values of ρ the curves show size dependence upto $L = 128$. But for $\rho < 0.8$ the curves become more and more flat with increasing system size. The mean load obeys a scaling form with $p_c - p$ and L of the following form (Fig. 5),

$$\bar{q}(\rho, t) L^{0.13} = \alpha \exp[-\beta(p_c - p)] \tag{2}$$

The value of α is 1.85 and β is 3.2.

Now just as p_c varies with ρ, the quantity ρ_c also varies with p (Fig. 2). We have studied the behaviour of the mean load as a function of ρ for a given value of p (Fig. 6). It is clear from the figure that as the value of ρ decreases the mean load becomes very small irrespective of the value of p. Moreover, for a given p, when ρ is less than ρ_c, the system is in a free-flow phase and the rate of flow increases with an increase of posting

rate. However, as the posting rate exceeds ρ_c, the rate of flow drops suddenly and the system is jammed. This effect is called *hysteresis* in the context of traffic flow [13]. For all values of p the mean load diverges algebraically with $\rho_c - \rho$,

$$N(\rho, t)/L^2 \propto (\rho_c - \rho)^{-\lambda} \tag{3}$$

The exponent λ is 1.0 for $p = 0.5$ as observed in [12] but decreases for p above and below 0.5 (Fig. 7). Thus it decreases to 0.49 at $p = 0.4$ and then remains constant even if p is decreased further. Again if p is increased above 0.5 the exponent decreases very rapidly to 0.28 at $p = 0.6$.

The fluctuation of mean load as a function of $\rho_c - \rho$ for different values of p exhibits almost a similar pattern like the mean load. The fluctuation is maximum for $p = 0.5$. As p is gradually increased or decreased from 0.5 the fluctuation decreases. It is probably due to the fact that when $p = 0.5$ we are selecting the recipient node randomly, whereas in other cases we are preferring a certain type of node (more populated or less populated) each time we forward a data.

To gain further insight into the change of the behaviour of the mean load as a function of p and ρ, we have studied the *queue length distribution* function $q(l)$ which is the distribution of the number of data packets (l) present at different sites. At any posting rate below ρ_c, as the system is always jammed for $p > p_c$, we shall confine our study of queue length distribution only in the free-flow phase, i.e. for $p < p_c$. At $p = 0.5$ the queue length distribution is exponential for all values of ρ in accordance with the result of Mukherjee and Manna. According to our study the queue length distribution is exponential for any value of $p < p_c$ above a certain value of ρ (dependent on system size). But below this value of ρ, the queue length distribution is algebraic (Fig. 8) only for a very narrow range of value of p (shaded region in Fig. 2). Below this narrow range of value of p the queue length distribution is exponential (Fig. 9).

In summary, we have studied the effect of introducing a probability p for choosing the more populated node (for onward data transfer) in a traffic network. For a posting rate below a lower critical value ρ_0, the system is in a free-flow phase for any p, but for $\rho > \rho_0$ the system undergoes a transition from a free-flow phase to a jammed phase as p exceeds

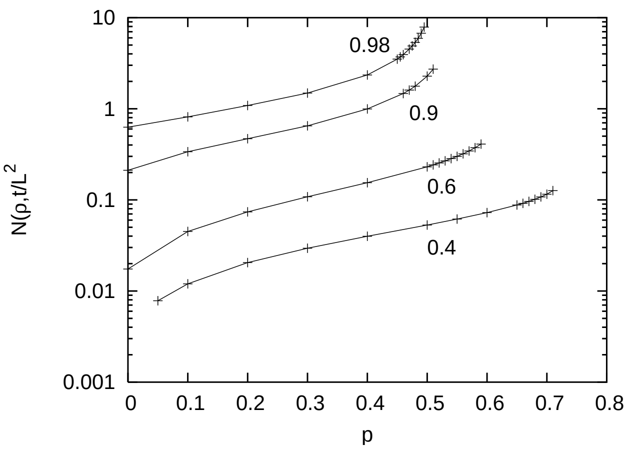

Figure 1: Plot of mean load $N(t)/L^2$ as a function of p for different values of ρ.

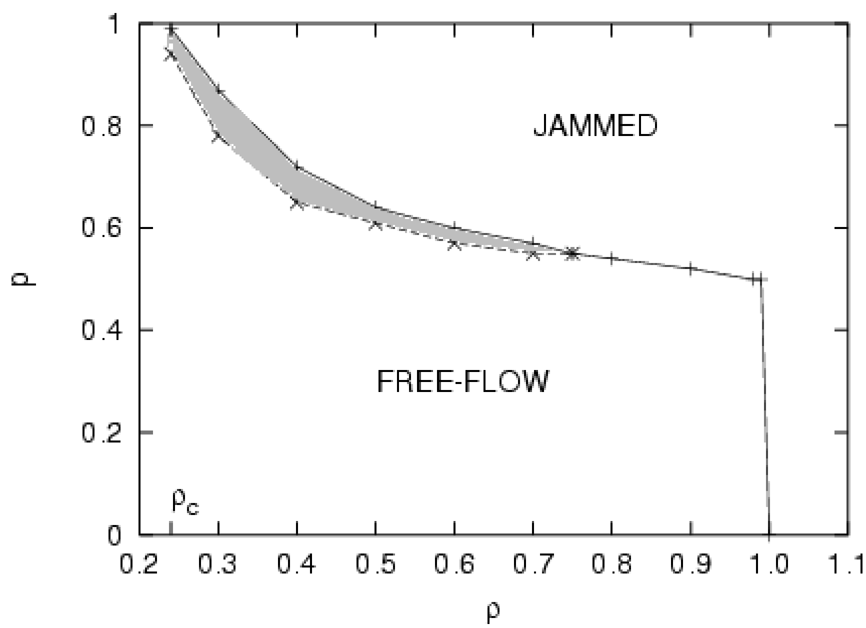

Figure 2: Various phases of the network for different values of ρ and p. The solid line is the boundary between the jammed and the free-flow phase and also the locus of $(\rho_c,\ p_c)$. The broken line separates the regions of algebraic and exponential queue-length distribution (both) within the free-flow phase. The algebraic nature of $q(l)$ exists only in the shaded region.

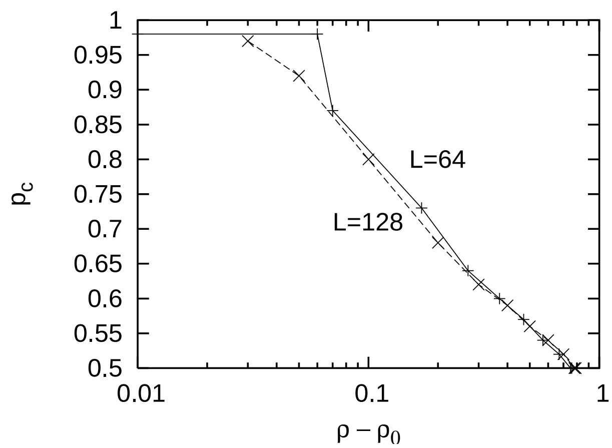

Figure 3: Plot of critical value p_c as a function of $\rho - \rho_0$ for system sizes $L = 64$ and $L = 128$. The curves fit roughly to $-0.155 \log(\rho - \rho_0) + 0.45$

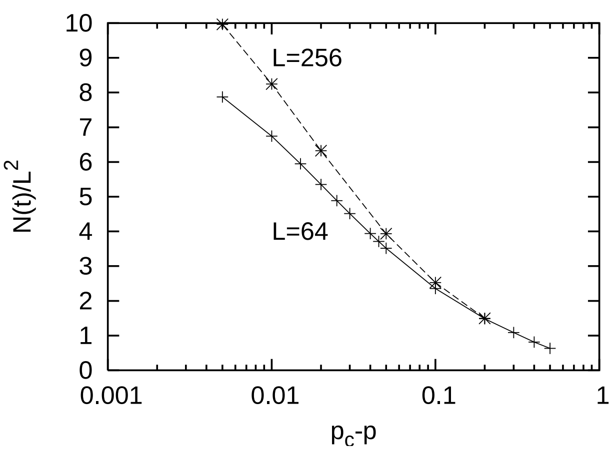

Figure 4: Plot of average load per site $N(t)/L^2$ as a function of $p_c - p$ for $\rho = 0.98$ for system sizes $L = 64$, 128 and 256. The linear portion of the curves fit to $-1.9 \log(p_c - p) - 2.16$ for $L = 64$ and to $-2.58 \log(p_c - p) - 3.75$ for $L = 256$. The curve for $L = 128$ coincides with $L = 256$

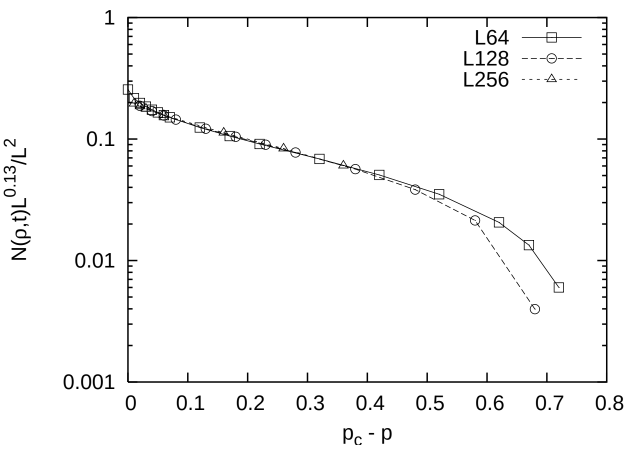

Figure 5: Scaled plot of the average load per site $N(t)/L^2$ as a function of $p_c - p$ for $\rho = 0.40$ and system sizes $L = 64$, 128 and 256. The scaled plot fits nicely in a normal-log graph.

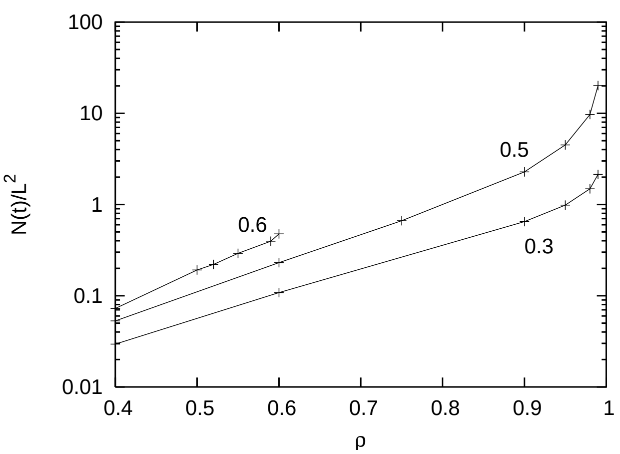

Figure 6: Plot of mean load $N(t)/L^2$ as a function of ρ for different values of p. The numbers labelling the curves correspond to the various values of p.

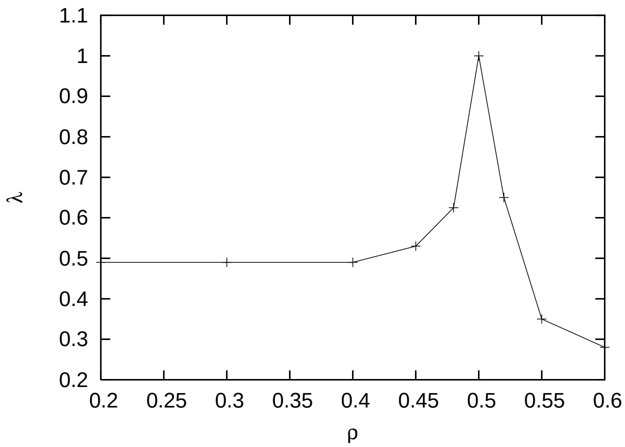

Figure 7: Exponent (λ) of mean load for different values of ρ (see Eq. 3). The exponent is maximum for $\rho = 0.5$.

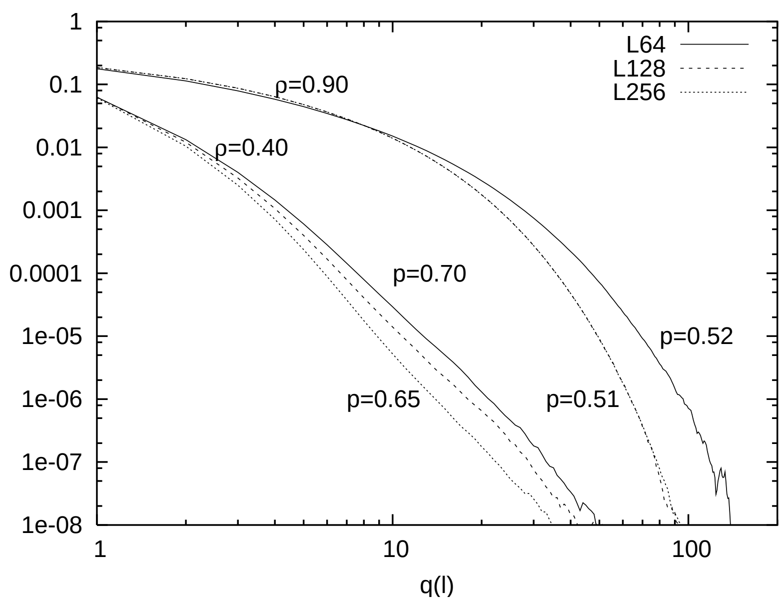

Figure 8: Queue length distribution for $\rho = 0.40$ and $\rho = 0.90$ for $L = 64$, 128 and 256 in log-log scale. The values of corresponding p values are shown beside the plots. For $\rho = 0.40$ the distribution is algebraic near $p = p_c$.

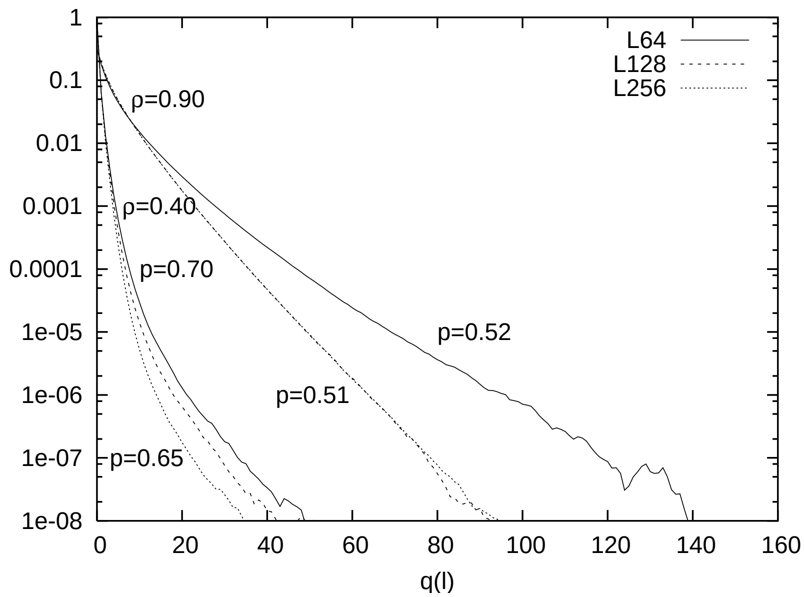

Figure 9: Queue length distribution for $\rho = 0.40$ and $\rho = 0.90$ for $L = 64$, 128 and 256 in normal-log scale. For $\rho = 0.90$ the distribution is exponential for any value of p less than p_c.

a critical value $p_c(< 1)$. As ρ increases from ρ_0 to the critical value ρ_c, the value of p_c also decreases from 1 to 0.

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